Lesson 7.2 Exercises, pages 592–599

A

Use algebra to solve each equation. Give exact values when possible; otherwise write the roots to the nearest degree or the nearest hundredth of a radian. Verify the solutions.

4. Solve each equation over the domain $0 \leq x < 2\pi$.

a) $\sin x = \frac{\sqrt{3}}{2}$

The reference angle is:

$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

In Quadrant 1, $x = \frac{\pi}{3}$

In Quadrant 2, $x = \pi - \frac{\pi}{3}$, or $\frac{2\pi}{3}$

b) $\tan x = \frac{1}{\sqrt{3}}$

The reference angle is:

$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

In Quadrant 1, $x = \frac{\pi}{6}$

In Quadrant 3, $x = \pi + \frac{\pi}{6}$, or $\frac{7\pi}{6}$

5. Verify that each given value of x is a root of the equation.

a) $\tan^2 x - 3 = 0;$
$x = \frac{\pi}{3}$

Substitute $x = \frac{\pi}{3}$ in each side of the equation.

$$\text{L.S.} = \tan^2\left(\frac{\pi}{3}\right) - 3$$
$$= (\sqrt{3})^2 - 3$$
$$= 0$$
$$= \text{R.S.}$$

Since the left side is equal to the right side, the root is verified.

b) $8 \sin^2 x + 6 \sin x + 1 = 0;$
$x = \frac{7\pi}{6}$

Substitute $x = \frac{7\pi}{6}$ in each side of the equation.

$$\text{L.S.} = 8 \sin^2\left(\frac{7\pi}{6}\right) + 6 \sin \frac{7\pi}{6} + 1$$
$$= 8\left(-\frac{1}{2}\right)^2 + 6\left(-\frac{1}{2}\right) + 1$$
$$= 2 - 3 + 1$$
$$= 0$$
$$= \text{R.S.}$$

Since the left side is equal to the right side, the root is verified.

B

6. Solve each equation over the domain $0 \leq x < 2\pi$, then state the general solution.

a) $3 \cos x - 2 = 0$

$$3 \cos x = 2$$
$$\cos x = \frac{2}{3}$$

In Quadrant 1,

$$x = \cos^{-1}\left(\frac{2}{3}\right)$$
$$x = 0.8410...$$

In Quadrant 4,

$$x = 2\pi - 0.8410...$$
$$x = 5.4421...$$

The roots are: $x \doteq 0.84$ and $x \doteq 5.44$

The general solution is:

$x \doteq 0.84 + 2\pi k, k \in \mathbb{Z}$ or
$x \doteq 5.44 + 2\pi k, k \in \mathbb{Z}$

b) $2 \tan x + \sqrt{5} = 0$

$$2 \tan x = -\sqrt{5}$$
$$\tan x = -\frac{\sqrt{5}}{2}$$

The reference angle is:

$$\tan^{-1}\left(\frac{\sqrt{5}}{2}\right) = 0.8410...$$

In Quadrant 2,

$$x = \pi - 0.8410...$$
$$x = 2.3005...$$

In Quadrant 4,

$$x = 2\pi - 0.8410...$$
$$x = 5.4421...$$

The roots are: $x \doteq 2.30$ and $x \doteq 5.44$

The general solution is:

$x \doteq 2.30 + \pi k, k \in \mathbb{Z}$ or
$x \doteq 5.44 + \pi k, k \in \mathbb{Z}$

7. Solve each equation for $-\pi \leq x \leq \pi$.

a) $3 \tan x - 3 = 5 \tan x - 1$

$2 \tan x = -2$

$\tan x = -1$

The reference angle is:

$\tan^{-1}(1) = \frac{\pi}{4}$

In Quadrant 2,

$x = \pi - \frac{\pi}{4}$

$x = \frac{3\pi}{4}$

In Quadrant 4,

$x = -\frac{\pi}{4}$

The roots are: $x = \frac{3\pi}{4}$ and

$x = -\frac{\pi}{4}$

b) $5(1 + 2 \sin x) = 2 \sin x + 1$

$5 + 10 \sin x = 2 \sin x + 1$

$8 \sin x = -4$

$\sin x = -\frac{1}{2}$

The reference angle is:

$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

In Quadrant 3,

$x = -\pi + \frac{\pi}{6}$

$x = -\frac{5\pi}{6}$

In Quadrant 4,

$x = -\frac{\pi}{6}$

The roots are: $x = -\frac{\pi}{6}$ and

$x = -\frac{5\pi}{6}$

8. a) Solve each equation for $-180° \leq x \leq 90°$.

i) $2 \csc x = 6$

$\csc x = 3$

$\sin x = \frac{1}{3}$

The reference angle is:

$\sin^{-1}\left(\frac{1}{3}\right) = 19.4712...°$

In Quadrant 1, $x = 19.4712...°$

Quadrant 2 is not in the domain.

The root is: $x \doteq 19°$

ii) $-6 = 3 \cot x$

$-2 = \cot x$

$\tan x = -\frac{1}{2}$

The reference angle is:

$\tan^{-1}\left(\frac{1}{2}\right) = 26.5650...°$

Quadrant 2 is not in the domain.

In Quadrant 4, $x = -26.5650...°$

The root is: $x \doteq -27°$

b) Solve each equation for $-90° \leq x \leq 180°$.

i) $4 \sec x = -5$

$\sec x = \frac{-5}{4}$

$\cos x = -\frac{4}{5}$

The reference angle is:

$\cos^{-1}\left(\frac{4}{5}\right) = 36.8698...°$

In Quadrant 2,

$x = 180° - 36.8698...°$

$x = 143.1301...°$

There is no solution in Quadrant 3.

The root is: $x \doteq 143°$

ii) $-\frac{1}{2} = \frac{1}{3} \csc x$

$\csc x = -\frac{3}{2}$

$\sin x = -\frac{2}{3}$

The reference angle is:

$\sin^{-1}\left(\frac{2}{3}\right) = 41.8103...°$

There is no solution in Quadrant 3.

In Quadrant 4,

$x = -41.8103...°$

The root is: $x \doteq -42°$

9. For each equation, determine the general solution over the set of real numbers, then list the roots over the domain $-\pi \leq x < 0$.

a) $\cos 3x - 1 = 5 \cos 3x + 2$

$-3 = 4\cos 3x$

$\cos 3x = -\frac{3}{4}$

$3x = \cos^{-1}\left(-\frac{3}{4}\right)$

The terminal arm of angle $3x$ lies in Quadrant 2 or 3.

The reference angle for angle $3x$ is:

$\cos^{-1}\left(\frac{3}{4}\right) = 0.7227\ldots$

In Quadrant 2,

$3x = \pi - 0.7227\ldots$

$x = 0.8062\ldots$

In Quadrant 3,

$3x = \pi + 0.7227\ldots$

$x = 1.2881\ldots$

The period of $\cos 3x$ is $\frac{2\pi}{3}$, so the general solution is:

$x \doteq 0.81 + \frac{2\pi}{3}k, k \in \mathbb{Z}$, or

$x \doteq 1.29 + \frac{2\pi}{3}k, k \in \mathbb{Z}$

In the given domain, the roots are:

$x = 0.8062\ldots - \frac{2\pi}{3}$

$x \doteq -1.29$, and

$x = 1.2881\ldots - \frac{2\pi}{3}$

$x \doteq -0.81$, and

$x = 1.2881\ldots - \frac{4\pi}{3}$

$x \doteq -2.90$

b) $3 \sin 4x = 3 - 2 \sin 4x$

$5\sin 4x = 3$

$\sin 4x = \frac{3}{5}$

$4x = \sin^{-1}\left(\frac{3}{5}\right)$

The terminal arm of angle $4x$ lies in Quadrant 1 or 2.

The reference angle for angle $4x$ is:

$\sin^{-1}\left(\frac{3}{5}\right) = 0.6435\ldots$

In Quadrant 1,

$4x = 0.6435\ldots$

$x = 0.1608\ldots$

In Quadrant 2,

$4x = \pi - 0.6435\ldots$

$x = 0.6245\ldots$

The period of $\sin 4x$ is $\frac{2\pi}{4}$, or $\frac{\pi}{2}$, so the general solution is:

$x \doteq 0.16 + \frac{\pi}{2}k, k \in \mathbb{Z}$, or

$x \doteq 0.62 + \frac{\pi}{2}k, k \in \mathbb{Z}$

In the given domain, the roots are:

$x = 0.1608\ldots - \frac{\pi}{2}$

$x \doteq -1.41$, and

$x = 0.1608\ldots - \pi$

$x \doteq -2.98$, and

$x = 0.6245\ldots - \frac{\pi}{2}$

$x \doteq -0.95$, and

$x = 0.6245\ldots - \pi$

$x \doteq -2.52$

10. Two students determined the general solution of the equation $3 \sin x + 5 = 5(\sin x + 1)$. Joseph said the solution is $x = 2\pi k$ or $x = \pi + 2\pi k$, where k is an integer. Yeoun Sun said the solution is $x = \pi k$, where k is an integer. Who is correct? Explain.

Both students are correct because both expressions produce the same roots: $x = 0, x = \pm\pi, x = \pm 2\pi, x = \pm 3\pi$, and so on

11. Solve each equation over the domain $-\pi \leq x \leq \frac{\pi}{2}$.

a) $4\cos^2 x - 3 = 0$

$4\cos^2 x = 3$

$\cos^2 x = \frac{3}{4}$

$\cos x = \pm\frac{\sqrt{3}}{2}$

The reference angle is:

$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

In Quadrant 1, $x = \frac{\pi}{6}$

In Quadrant 2, there is no solution in the given domain.

In Quadrant 3, $x = -\pi + \frac{\pi}{6}$

$x = -\frac{5\pi}{6}$

In Quadrant 4, $x = -\frac{\pi}{6}$

The roots are: $x = \pm\frac{\pi}{6}$ and $x = -\frac{5\pi}{6}$

b) $2\tan^2 x = 3$

$\tan^2 x = \frac{3}{2}$

$\tan x = \pm\sqrt{\frac{3}{2}}$

The reference angle is:

$\tan^{-1}\left(\sqrt{\frac{3}{2}}\right) = 0.8860\ldots$

In Quadrant 1, $x = 0.8860\ldots$

In Quadrant 2, there is no solution in the given domain.

In Quadrant 3, $x = -\pi + 0.8860\ldots$

$x = -2.2555\ldots$

In Quadrant 4, $x = -0.8860\ldots$

The roots are: $x \doteq \pm 0.89$ and $x \doteq -2.26$

12. Use factoring to solve each equation over the domain $-90° \leq x < 270°$.

a) $2\cos x \sin x - \cos x = 0$

$(\cos x)(2\sin x - 1) = 0$

Either $\cos x = 0$

$x = \pm 90°$

Or $2\sin x - 1 = 0$

$\sin x = 0.5$

$x = 30°$

or $x = 180° - 30°$

$x = 150°$

The roots are: $x = 30°$, $x = \pm 90°$, and $x = 150°$

b) $3\tan x + \tan^2 x = 2\tan x$

$\tan x + \tan^2 x = 0$

$(\tan x)(1 + \tan x) = 0$

Either $\tan x = 0$

$x = 0°$ or $x = 180°$

Or $1 + \tan x = 0$

$\tan x = -1$

$x = 135°$ or $x = -45°$

The roots are: $x = 0°$, $x = 180°$, $x = 135°$, and $x = -45°$

13. A student wrote the solution below to solve the equation $2\sin^2 x + \sin x = 1$ over the domain $0 \leq x < 2\pi$. Identify any errors, then write a correct solution.

$2\sin^2 x + \sin x = 1$

$(\sin x)(2\sin x + 1) = 1$

$\sin x = 1$ or $2\sin x + 1 = 1$

$x = \frac{\pi}{2}$ $\quad$ $\sin x = 0$

$x = 0$ or $x = \pi$

$2\sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0$

Either $2\sin x - 1 = 0$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$

Or $\sin x + 1 = 0$

$\sin x = -1$

$x = \frac{3\pi}{2}$

There is an error in the second line of the solution. One side of the equation must be 0 before factoring, so collect all the terms on the left side.

14. Solve each equation over the domain $-2\pi \leq x \leq 2\pi$, then determine the general solution.

a) $2\cos^2 x - \cos x - 1 = 0$

$(2\cos x + 1)(\cos x - 1) = 0$
For $2\cos x + 1 = 0$
$2\cos x = -1$
$\cos x = -\frac{1}{2}$
The terminal arm of angle x lies in Quadrant 2 or 3.
The reference angle is:
$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$
In Quadrant 2,
$x = \pi - \frac{\pi}{3}$, or $\frac{2\pi}{3}$
and, $x = -\pi - \frac{\pi}{3}$, or $-\frac{4\pi}{3}$
In Quadrant 3,
$x = \pi + \frac{\pi}{3}$, or $\frac{4\pi}{3}$
and, $x = -\pi + \frac{\pi}{3}$, or $-\frac{2\pi}{3}$
For $\cos x - 1 = 0$
$\cos x = 1$
The terminal arm lies along the positive x-axis.
So, $x = \pm 2\pi$ or $x = 0$
The roots are: $x = 0$, $x = \pm\frac{2\pi}{3}$, $x = \pm\frac{4\pi}{3}$, and $x = \pm 2\pi$
Since consecutive roots differ by $\frac{2\pi}{3}$, the general solution is:
$x = \frac{2\pi}{3}k, k \in \mathbb{Z}$

b) $5\sin^2 x + 3\sin x = 2$

$5\sin^2 x + 3\sin x - 2 = 0$
$(5\sin x - 2)(\sin x + 1) = 0$
For $5\sin x - 2 = 0$
$5\sin x = 2$
$\sin x = \frac{2}{5}$
The terminal arm of angle x lies in Quadrant 1 or 2.
The reference angle is:
$\sin^{-1}\left(\frac{2}{5}\right) = 0.4115...$
In Quadrant 1,
$x = 0.4115...$
and, $x = -2\pi + 0.4115...$
$x = -5.8716...$
In Quadrant 2,
$x = \pi - 0.4115...$
$x = 2.7300...$
and, $x = -\pi - 0.4115...$
$x = -3.5531...$
For $\sin x + 1 = 0$
$\sin x = -1$
The terminal arm lies along the negative y-axis.
So, $x = \frac{3\pi}{2}$ or $x = -\frac{\pi}{2}$
The roots are: $x \doteq -5.87$, $x \doteq -3.55$, $x = -\frac{\pi}{2}$, $x \doteq 0.41$, $x \doteq 2.73$, and $x = \frac{3\pi}{2}$
Since the period is 2π, the general solution is:
$x \doteq 0.41 + 2\pi k, k \in \mathbb{Z}$ or
$x \doteq 2.73 + 2\pi k, k \in \mathbb{Z}$ or
$x = \frac{3\pi}{2} + 2\pi k, k \in \mathbb{Z}$

15. Solve each equation over the domain $0 \leq x < 2\pi$.

a) $4\tan^2 x = 2 - 5\tan x$

$4\tan^2 x + 5\tan x - 2 = 0$

Use: $\tan x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Substitute: $a = 4$, $b = 5$, $c = -2$

$$\tan x = \frac{-5 \pm \sqrt{5^2 - 4(4)(-2)}}{2(4)}$$

$$\tan x = \frac{-5 \pm \sqrt{57}}{8}$$

For $\tan x = \dfrac{-5 + \sqrt{57}}{8}$

$\tan x$ is positive when the terminal arm of angle x lies in Quadrant 1 or 3.

The reference angle is:

$$\tan^{-1}\left(\frac{-5 + \sqrt{57}}{8}\right) = 0.3085\ldots$$

In Quadrant 1, $x = 0.3085\ldots$

In Quadrant 3,

$x = \pi + 0.3085\ldots$

$x = 3.4501\ldots$

For $\tan x = \dfrac{-5 - \sqrt{57}}{8}$

$\tan x$ is negative when the terminal arm of angle x lies in Quadrant 2 or 4.

The reference angle is:

$$\tan^{-1}\left(\frac{5 + \sqrt{57}}{8}\right) = 1.0032\ldots$$

In Quadrant 2,

$x = \pi - 1.0032\ldots$

$x = 2.1383\ldots$

In Quadrant 4,

$x = 2\pi - 1.0032\ldots$

$x = 5.2798\ldots$

The roots are: $x \doteq 0.31$, $x \doteq 2.14$, $x \doteq 3.45$, and $x \doteq 5.28$

b) $4\sin x + 3 = 2\sin^2 x$

$2\sin^2 x - 4\sin x - 3 = 0$

Use: $\sin x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Substitute: $a = 2$, $b = -4$, $c = -3$

$$\sin x = \frac{4 \pm \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)}$$

$$\sin x = \frac{4 \pm \sqrt{40}}{4}, \text{ or } \frac{2 \pm \sqrt{10}}{2}$$

For $\sin x = \dfrac{2 + \sqrt{10}}{2}$; this is greater than 1, so there is no real solution.

For $\sin x = \dfrac{2 - \sqrt{10}}{2}$

$\sin x$ is negative when the terminal arm of angle x lies in Quadrant 3 or 4.

The reference angle is:

$$\sin^{-1}\left(\frac{\sqrt{10} - 2}{2}\right) = 0.6201\ldots$$

In Quadrant 3,

$x = \pi + 0.6201\ldots$

$x = 3.7617\ldots$

In Quadrant 4,

$x = 2\pi - 0.6201\ldots$

$x = 5.6630\ldots$

The roots are: $x \doteq 3.76$ and $x \doteq 5.66$

C

16. Write a second-degree trigonometric equation that has roots $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ over the domain $0 \leq x < 2\pi$.

Sample response: Since $\sin \frac{\pi}{6} = \sin \frac{5\pi}{6}$, these two roots can be determined from one factor, so use the sine ratio. Work backward.

Either $x = \frac{\pi}{6}$ or $x = \frac{\pi}{2}$

So, $\sin x = \frac{1}{2}$ or $\sin x = 1$

Then, $\sin x - \frac{1}{2} = 0$ or $\sin x - 1 = 0$

So, an equation is: $\left(\sin x - \frac{1}{2}\right)(\sin x - 1) = 0$

This can be written as: $(2 \sin x - 1)(\sin x - 1) = 0$ or $2 \sin^2 x - 3 \sin x + 1 = 0$

17. Determine the number of roots each equation has over the domain $0 \leq x < 2\pi$, where $1 < a < b$.

a) $(a \cos x - b)(b \sin x + a) = 0$

Solve the equation.
Either $a \cos x - b = 0$

$$\cos x = \frac{b}{a}$$

But $b > a$, so $\frac{b}{a} > 1$, and there is no real solution
Or $b \sin x + a = 0$

$$\sin x = -\frac{a}{b}$$

Since $b > a$, there is a root in the two quadrants where the sine ratio is negative.
So, there are 2 roots.

b) $(b \sin^2 x - 1)(a \tan x + b) = 0$

Solve the equation.
Either $b \sin^2 x - 1 = 0$

$$\sin x = \pm\sqrt{\frac{1}{b}}$$

Since $b > 1$, $0 < \sqrt{\frac{1}{b}} < 1$, and there is a root in each quadrant.
Or $a \tan x + b = 0$

$$\tan x = -\frac{b}{a}$$

There is a root in the two quadrants where the tangent ratio is negative.
So, there is a total of 6 roots.